B.SC. THIRD SEMESTER (HONOURS) EXAMINATIONS, 2021

Subject: Mathematics **Course ID: 32111**

Course Code: SH/MTH/301/C-5

Course Title: Theory of Real Functions and Introduction to Metric Space

Full Marks: 40 **Time: 2 Hours**

The figures in the margin indicate full marks

Notations and symbols have their usual meaning

1. Answer *any five* of the following questions: (2 x 5= 10)

a) Let X be a non-empty set and $f: X \to R$ be an injective function. Then prove that $d(x, y) = |f(x) - f(y)|, \forall\, x, y \in X$ defines a metric on X.

b) If $f(x) = (x^2 - 1)|x^2 - 3x + 2| + \cos|2x|$, then find the set of points of non differentiability.

c) If $f(x)$ is differentiable on (a, b) and $f(a) = 0$ and there exists a real number k such that $|f'(x)| \leq k|f(x)|$ on $[a, b]$, then show that $f(x) = 0$ for all $x \in [a, b]$.

d) If $f(x + y) = f(x)f(y)$ for all x and y, $f(3) = 3$ and $f'(0) = 11$, then find $f'(3)$.

e) If $P(x)$ is a polynomial of degree >1 and $k \in R$. Prove that between any two real roots of $P(x)$=0 there is a real root of $P'(x) + kP(x) = 0$.

f) Show that there does not exist a function $\emptyset$ such that $\emptyset'(x) = f(x)$, where $f(x) = [x]$.

g) Let (X, d) be a metric space and $Y \subseteq X$ where Y is separable and dense in X. Show that X is separable.

h) If $f(x) = sinx$, prove that $\lim_{h\to 0} \theta = \frac{1}{\sqrt{3}}$, where θ is given by $f(h) = f(0) + hf'(\theta h)$.

2. Answer *any four* of the following questions: (5 x 4 = 20)

a) (i) If the functions f, g and their derivatives are continuous in $[a, b]$ and if $f(x)g'(x) - g(x)f'(x) \neq 0$, then show that between any two roots of $f(x) = 0$ there lies a root of $g(x) = 0$.

(ii) Prove or disprove: "Let f be continuous in a bounded open interval (a, b) and $\lim_{x\to a+} f(x)$ and $\lim_{x\to b-} f(x)$ both exist finitely. Then $f(x)$ is uniformly continuous on $[a, b]$." 3+2

b) Let $a, b \in R$ and $a < b$. If a function f is continuous on (a, b) except possibly at $c \in (a, b)$. If $\lim_{x\to c} f'(x)$ is finite , then prove that $f'(c) = l$.

c) (i) Show that $'\theta'$ which occurs in the Lagrange's Mean Value theorem tends to $\frac{1}{2}$ as $h \to 0$, provided $f'''(x)$ is continuous.

(ii) Let $f\left(\frac{x+y}{2}\right) = \frac{1}{2}[f(x) + f(y)]$, for all x and y. If $f'(0)$ exists and equals (-1) and $f(0) = 1$, find $f(3)$. 3+2

d) Let $D \subset R$ be a bounded set and $f: D \to R$ be a function. If for each Cauchy sequence $\{x_n\}$ in D, the sequence $\{f(x_n)\}$ is a Cauchy sequence in R, then show that f is uniformly continuous on D.

e) (i) Expand $\sin x$ in powers of $\left(x - \frac{\pi}{2}\right)$ with the help of Taylor's theorem.

(ii) Let the real line R be endowed with usual metric *d*. Give an example of two closed subsets *A* and *B* in (R, d) such that $d(A, B) = 0$, but $A \cap B = \phi$. (3+2)

f) Let us define a function $d: R^2 \times R^2 \to R$ by

$$d(x, y) = \begin{cases} |x_2 - y_2| & if\ x_1 = y_1 \\ |x_1 - y_1| + |x_2\ | + |y_2| & if\ x_1\ \neq y_1 \end{cases}$$

where $x = (x_{1,}x_{2,}), y = (y_{1,}y_2)$ are arbitrary elements of R^2. Prove that d is a metric on R^2 .

3. Answer *any one* of the following questions: (10 x 1= 10)

a) (i) Let f be twice differentiable and $|f(x)| < \alpha, |f''(x)| < \beta$, for $x > a$, then show that $|f'(x)| < 2\sqrt{\alpha\beta}$, for all $x > a$.

(ii) If $x_{1,}x_2, \ldots, x_n$ are arbitrary points of a metric space (*X*, *d*), then prove that $d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n)$.

(iii) Let *X* be the set of all real-valued continuous functions on $[0,1]$. Define $d(f, g) = \int_0^1 |f(x) - g(x)| dx$, for all $f, g \in X$. Show that *d* is a metric for *X*. (3+4+3)

b) (i) Let $f: R \to R$ be continuous on R. A point $c \in R$ is said to be a fixed point of f if $f(c) = c$ holds. Prove that the set of all fixed points of f is a closed set.

(ii) If *f* is defined and differentiable on an interval, then show that the range of *f′* is an interval

(iii) Let $I = [a, b]$ be a closed and bounded interval, and a function $f: I \to R$ be continuous on I. Then show that f is uniformly continuous on I. (3+3+4)

***